Economics of Risk

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Market and Finance

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Uncertainty

Uncertainty is an essential characteristic of our economic lives

This is also true (and particularly) for things related to Green Energy Markets and Finance

Examples ?

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Uncertainty – some economist's examples

Example 1: A farmer chooses to cultivate either apples or pears

- When he takes the decision, he is uncertain about the profits that he will obtain. He does not know which is the best choice
- This will depend on rain conditions, plagues, world prices…

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Uncertainty

Some theoretical / formal examples:

Example 2: playing with a fair die

- We will win **$** 2 if 1, 2, or 3,
- We neither win nor lose if 4, or 5
- We will lose **$** 6 if 6

Example 3: John's monthly consumption:

- **$** 3000 if he does not get ill
- **$** 500 if he gets ill (so he cannot work)

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Our objectives in this part

- Review how economists make predictions about individual's or firm's choices under uncertainty
- Review the standard assumptions about attitudes towards risk

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Economist's jargon

Economists call a lottery a situation which involves uncertain payoffs:

- Cultivating apples is a lottery
- Cultivating pears is another lottery
- Playing with a fair die is another one
- Monthly consumption

Each lottery will result in a prize

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Probability

The probability of a repetitive event happening is the relative frequency with which it will occur

- probability of obtaining a head on the fair-flip of a coin is 0.5

If a lottery offers *n* distinct prizes and the probabilities of winning the prizes are p_i (i=1,…,*n*) then

$$\sum_{i=1}^{n} p_i = p_1 + p_2 + \ldots + p_n = 1$$

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How to bet? The origins of probability

Let's go back to 1600s

- Chevalier de Mere was a nobleman who gambled frequently
- He bet on a roll of a die that at least one 6 would appear during a total of four rolls
- From past experience, he knew that he was more successful than not in this bet
- He bet he would get a double 6 on 24 rolls of two dice
- Soon, he realized that this bet was not as profitable

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How to bet? The origins of probability

- Then, de Mere started bet he would get a double 6 on 24 rolls of two dice
- Soon, he realized that this bet was not as protable
- He asked his friend Blaise Pascal why
- Pascal developed a correspondence with Pierre de Fermat, and they are both credited with the founding of probability theory

Key idea: a bet should be evaluated by its expected value.

What is Expected Value?

For a lottery (X) with prizes $x_1, x_2, \ldots, x_n$ and the probabilities of winning $p_1, p_2, \ldots p_n$, the expected value of the lottery is

$$E(X) = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n$$

$$E(X) = \sum_{i=1}^{n} p_i x_i$$

- The expected value is a weighted sum of the prizes
  - the weights the respective probabilities
- The symbol for the expected value of X is *E(X)*

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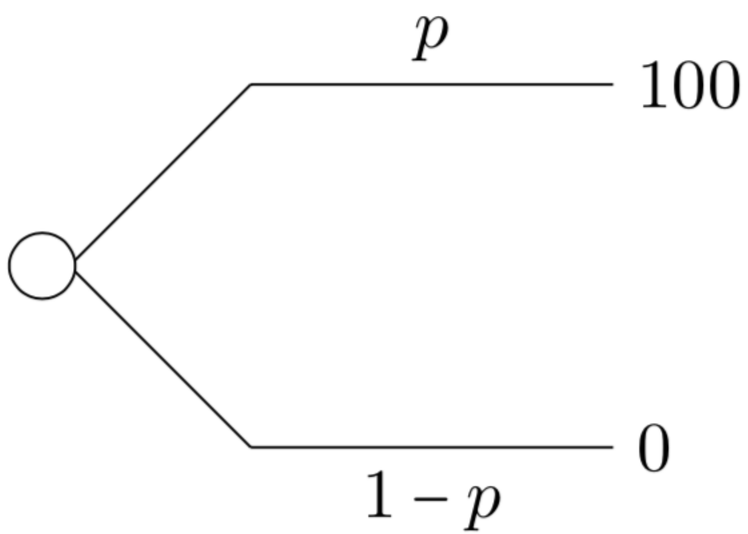
How to calculate Expected Value

- Consider a bet that pays $100 if some **event** happens, and nothing otherwise. It is represented below:



- What is its Expected Value?

$$E[X] = p \cdot 100 + (1 - p) \cdot 0 = 100p.$$

How to calculate Expected Value

- Suppose that the event is the first bet considered by de Mere, that is, to obtain a 6 if the dice is played four times. The probability of not getting 6 in four rolls is:

$$\left(\frac{5}{6}\right)^4 \approx 0.48 < \frac{1}{2}.$$

- Thus, the probability of getting at least a 6 is:

$$p = 1 - \left(\frac{5}{6}\right)^4 \approx 0.52 > \frac{1}{2}.$$

- The Expected Value of playing the bet of $\$100$ in this event is therefore

$$E[X] = p \cdot 100 + (1 - p) \cdot 0 \approx 52 > 50.$$

How to calculate Expected Value

- Now, assume the event is the second bet considered by de Mere, that is, to obtain a double 6 if the dice is played 24 times. The probability of not getting a double six in 24 rolls is:

$$\left(\frac{35}{36}\right)^{24} \approx 0.509 > \frac{1}{2}.$$

- Thus, the probability of winning the bet is:

$$p = 1 - \left(\frac{5}{6}\right)^{4} \approx 0.491 < \frac{1}{2}.$$

- The Expected Value of playing the bet of $\$100$ in this event is therefore

$$E[X] = p \cdot 100 + (1 - p) \cdot 0 \approx 49.1 < 50.$$

Another example –

The expected value of a lottery is the average of the prizes obtained if we play the same lottery many times

- If we played 600 times the lottery in Example 2
- We obtained a “1” 100 times, a “2” 100 times…
- We would win “ **$** 2” 300 times, win “ **$** 0” 200 times, and lose “ **$** 6” 100 times
- Average prize=(300* **$** 2+200* **$** 0-100* **$** 6)/600
- Average prize=(1/2)* **$** 2+(1/3)* **$** 0-(1/6)* **$** 6= **$** 0
- Notice, we have the probabilities of the prizes multiplied by the value of the prizes

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Expected Value of monthly consumption (Example 3)

Example 3: John's monthly consumption:

- X_1= **$** 4000 if he does not get ill
- X_2= **$** 500 if he gets ill (so he cannot work)
- Probability of illness 0.25
- Consequently, probability of no illness=1-0.25=0.75
- The expected value is:

$$E(X) = p_1 x_1 + p_2 x_2$$

$$E\,(X) = 0.75\ \cdot 4000 + 0.25\ \cdot 500 = 3125.$$

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Drawing the combinations of consumption with the same expected value

Only possible if we have at most 2 possible states (e.g. ill or not ill as in Example 3)

Given the probability p_1 then $p_2=1-p_1$

How can we graph the combinations of (X_1,X_2) with a expected value of, say, "E"?

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Drawing the combinations of consumption with the same expected value

The combinations of (X_1,X_2) with an expected value of, say, “E”?

$$E = p_1 x_1 + p_2 x_2 \quad \rightarrow \quad E = p_1 x_1 + (1 - p_1) x_2$$

$$x_2 = \frac{E}{1 - p_1} - \frac{p_1}{1 - p_1} x_1$$

$$if \quad x_1 = 0, then \quad x_2 = \frac{E}{1 - p_1}; if \quad x_2 = 0, then \quad x_1 = \frac{E}{p_1}$$

$$\frac{dx_2}{dx_1} = -\frac{p_1}{1 - p_1} \text{ (slope)}$$

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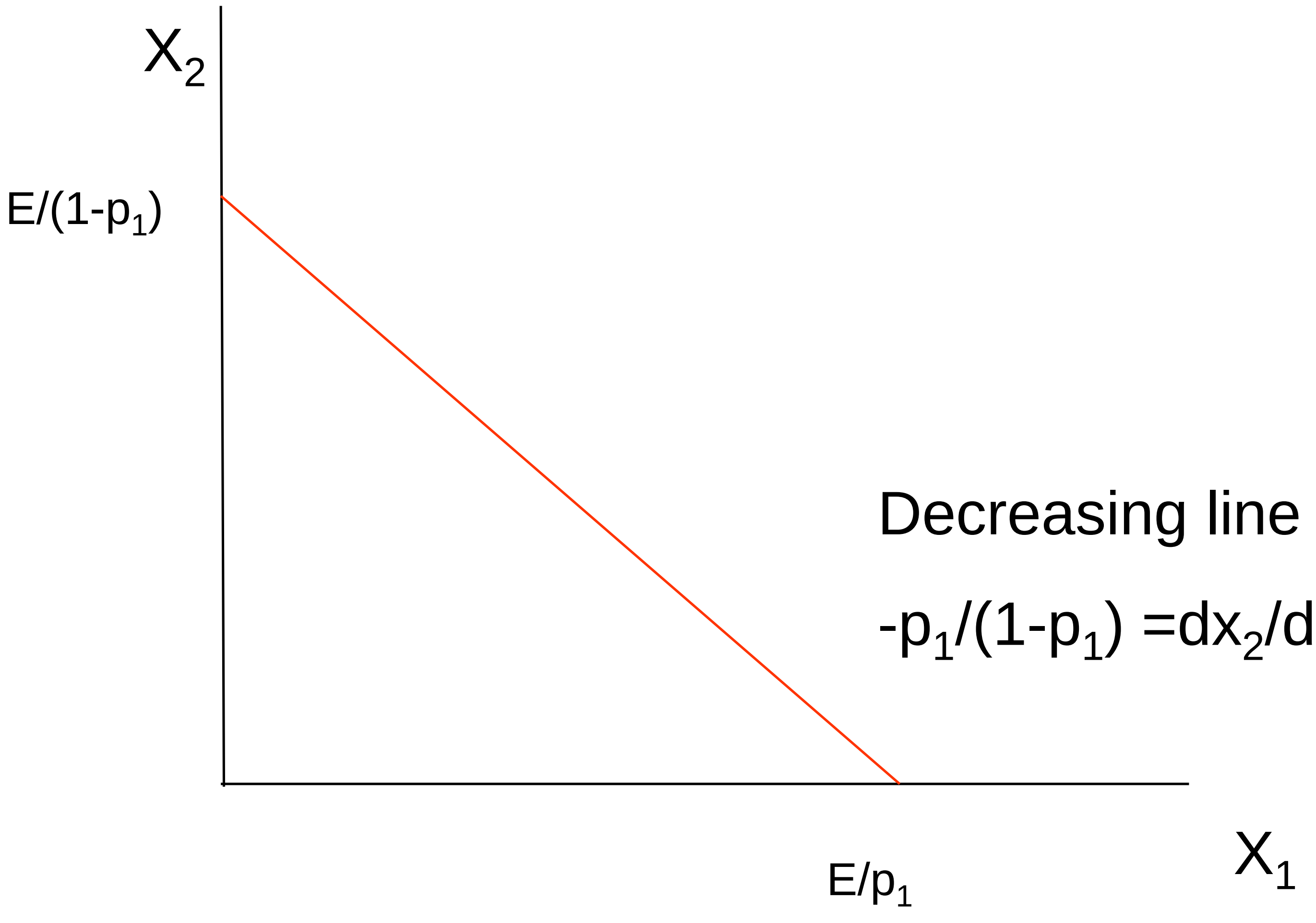
Drawing the combinations of consumption with the same expected value



Decreasing line with slope:

$-p_1/(1-p_1) = dx_2/dx_1$

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Introducing another lottery in John's example

- Lottery A: Get $ 3125 for sure independently of illness state (i.e. expected value= $ 3125). This is a lottery without risk

- Lottery B: win $ 4000 with probability 0.75,
- and win $ 500 with probability 0.25
- (i.e. expected value also $ 3125)

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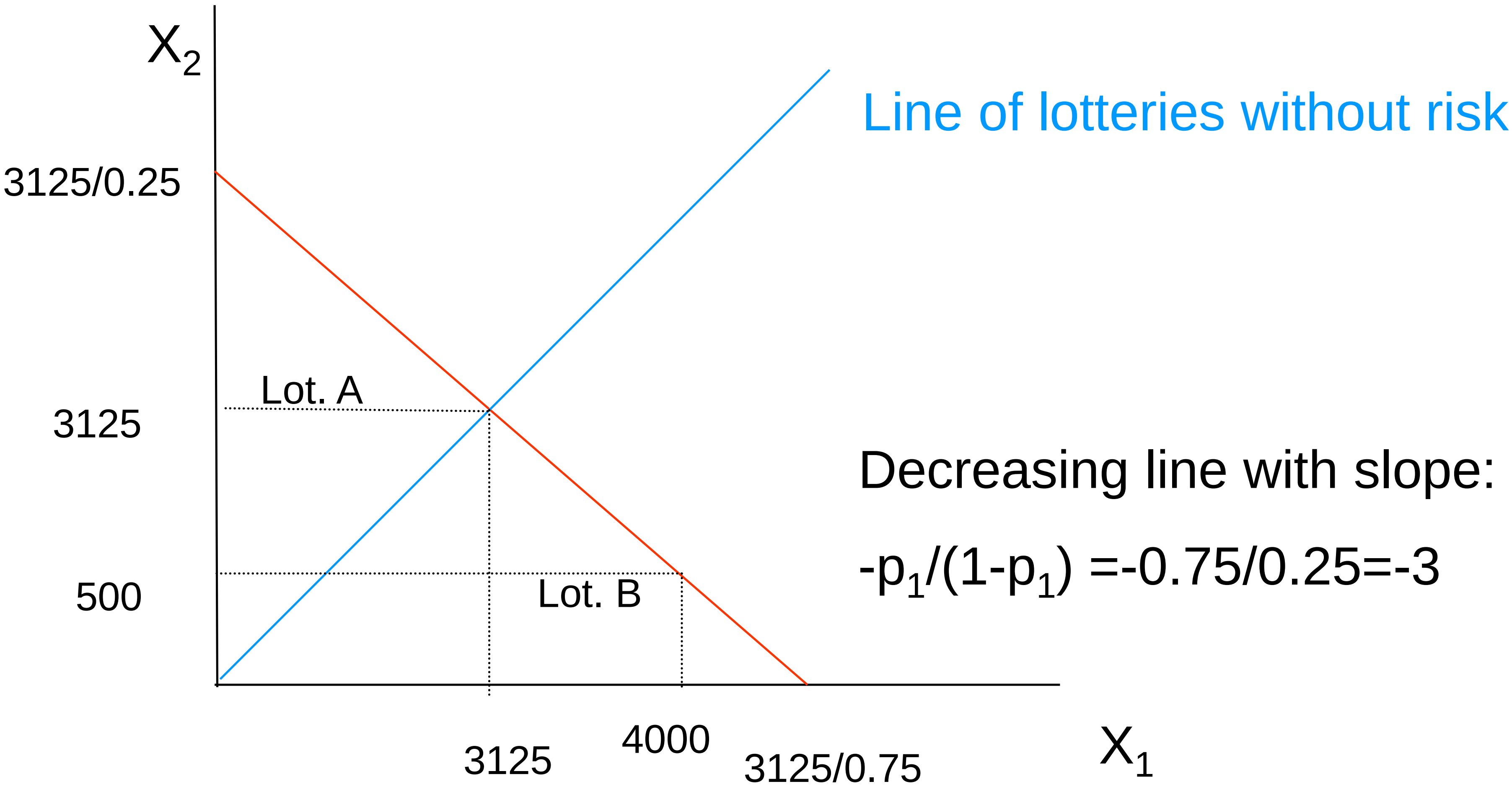
Drawing the combinations of consumption with the same expected value. Example 3



Decreasing line with slope:

$-p_1/(1-p_1) = -0.75/0.25 = -3$

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Risk versus Uncertainty

We have a situation with risk or uncertainty if there is some outcome of interest that may assume different values, depending on unknown factors

Modern economics differentiate risk and uncertainty as follows:

- We have a situation of ***risk*** if we know the *probabilities* of occurrence of each possible value of the relevant variable
- We have a situation of uncertainty if we do not know the actual probabilities of occurrence of each possible value of the relevant variable

We can calculate Expected Utility in the last case if we have subjective probabilities

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Is the *expected value* a good criterion to decide between lotteries?

One criterion to choose between two lotteries is to choose the one with a higher expected value

Does this criterion provide reasonable predictions? Let's examine a case…

- Lottery A: Get **$** 3125 for sure (i.e. expected value= **$** 3125)
- Lottery B: win **$** 4000 with probability 0.75,
- and lose **$** 500 with probability 0.25
- (i.e. expected value also **$** 3125)

What will you choose?

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Is the *expected value* a good criterion to decide between lotteries?

- Probably most people will choose Lottery A because they dislike risk (risk averse)
- However, according to the expected value criterion, both lotteries are equivalent. The expected value does not seem a good criterion for people that dislike risk
- If someone is indifferent between A and B it is because risk is not important for him (risk neutral)

- But let's see how this idea started…

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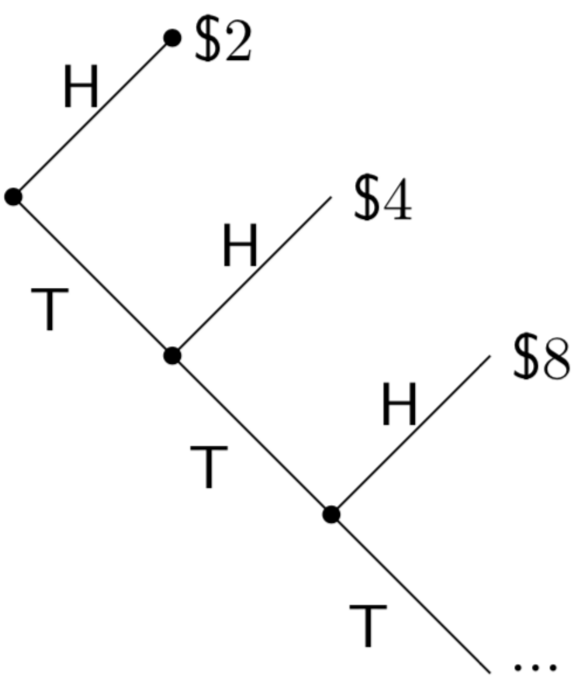
The birth of Expected Utility

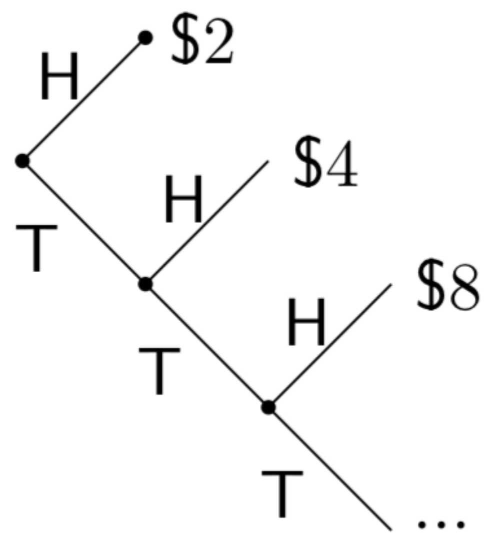
In 1713, Nicolas Bernoulli proposed the following bet:

- You toss a coin. In case of H, you receive $\$2$. In case of T, you toss it again.
- If H, you receive $\$4$. If T, you toss it again.
- In each play, the value is doubled.

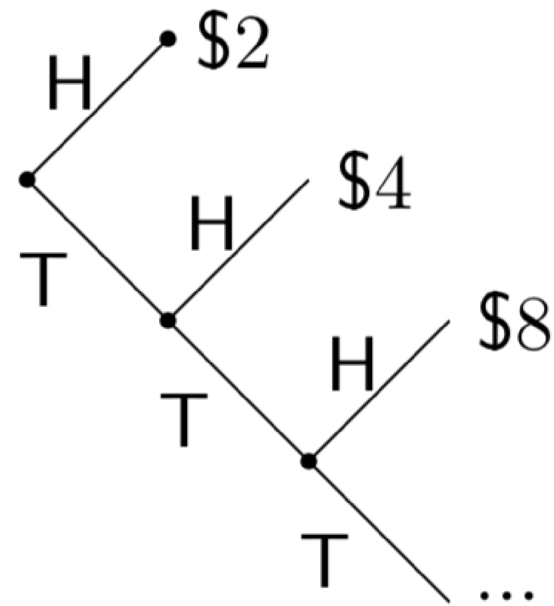


The birth of Expected Utility

- How much is this bet worth?
- How much would you pay to play it?



- How much is this bet worth?
- How much would you pay to play it?



$$
\begin{aligned}
\mathsf{E}[X] &= \frac{1}{2}2 + \frac{1}{4}4 + \frac{1}{8}8 + \frac{1}{16}16 + \ldots \\
&= 1 + 1 + 1 + 1 + \ldots \\
&= \infty!
\end{aligned}
$$

The birth of Expected Utility

- In 1738, Daniel Bernoulli (cousin of Nicolas Bernoulli), proposed a solution to the paradox:

  > *The determination of the value of an item must not be based on the price, but rather on the utility it yields... There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount.*

- Daniel Bernoulli lived in St. Petersburg when he published his solution—and this is how the paradox acquired its name

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Expected utility: The standard criterion to choose among lotteries

- Individuals do not care directly about the monetary values of the prizes
  - they care about the utility that the money provides
- *U(x)* denotes the utility function for money
- We will always assume that individuals prefer more money than less money, so *U* is increasing.
- If *U* is differentiable, this happens if:

$$U'(x_i) > 0$$

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Expected utility: The standard criterion to choose among lotteries

The expected utility is computed in a similar way to the expected value

However, one does not average prizes (money) but the utility derived from the prizes

The formula of expected utility is:

$$EU = \sum_{i=1}^{n} p_i U(x_i) = p_1 U(x_1) + p_2 U(x_2) + \ldots + p_n U(x_n)$$

- The individual will choose the lottery with the highest expected utility

Attitude towards risk: Risk aversion

We say that an agent who maximizes Expected Utility EU is risk averse if whenever this agent is offered the choice between two prospects X and Y, such that

$$X = E[\ Y\ ],$$

that is, X is risk free and equal to the expected value of Y,

then the agent always prefer the risk free prospect A, that is,

$$U(X) = E[\ U(X)\ ] \geq E[\ U(Y)\ ].$$

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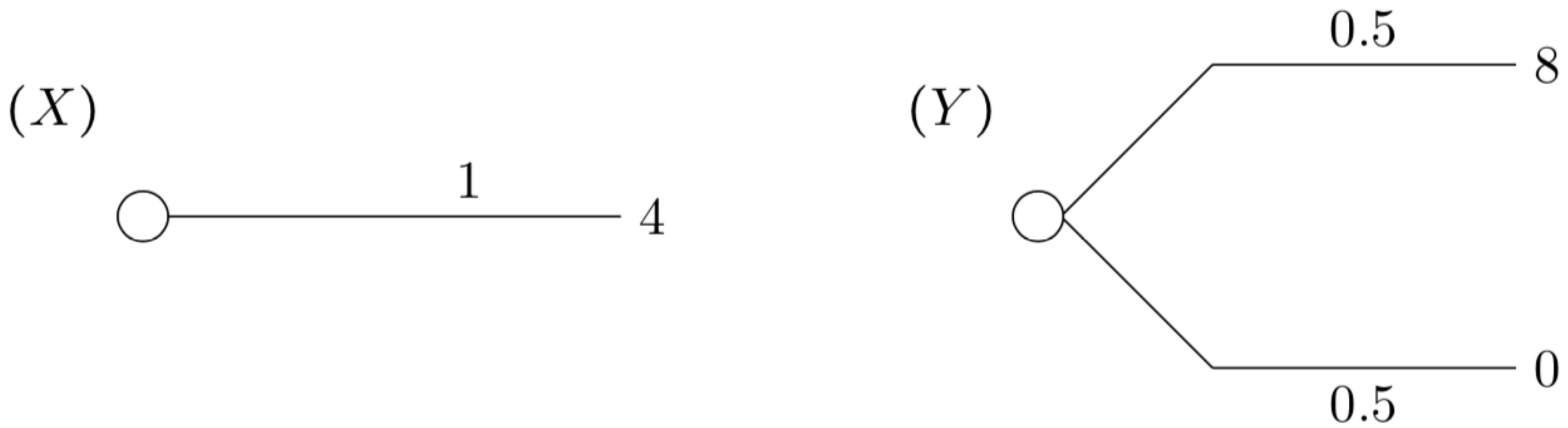
Example of risk aversion

Let X be equal to 4 (without risk) and let Y that pays 8 or 0, each with probability ½. See the illustration:



Notice that $E(Y) = \frac{1}{2} \cdot 8 + \frac{1}{2} \cdot 0 = 4 = E(X)$.

The Decision Maker (DM) is risk averse if he/she prefers *X*.

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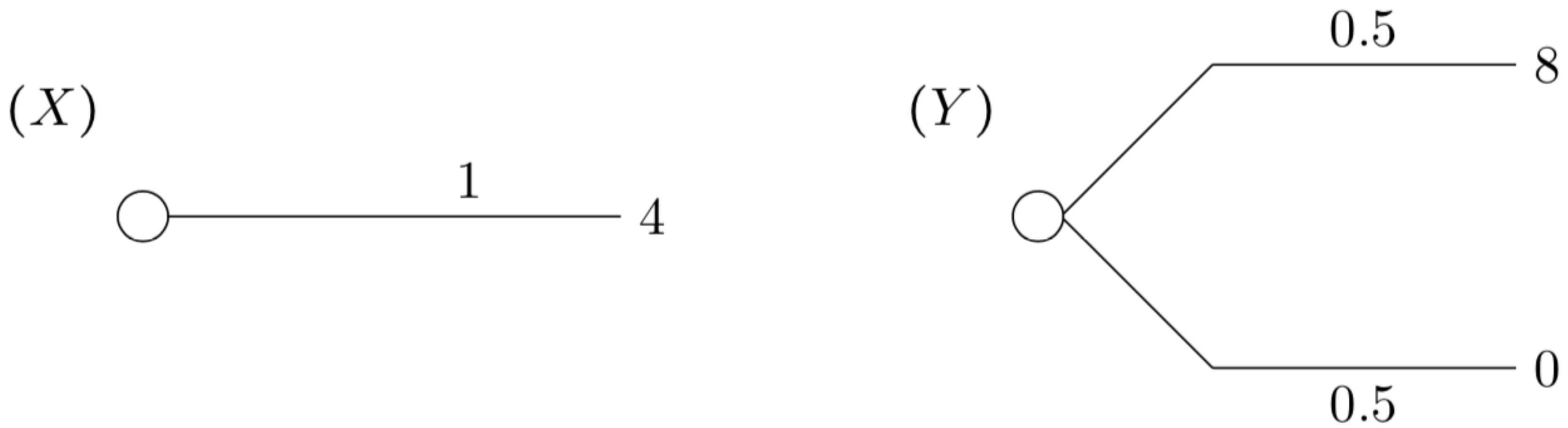
Characterization of risk aversion

$$E(Y) = \frac{1}{2} \cdot 8 + \frac{1}{2} \cdot 0 = 4 = E(X).$$



The Decision Maker (DM) is risk averse if he/she prefers *X, that is,*

$$E[U(X)] = U(4) \geq E[U(Y)] = \frac{1}{2} \cdot U(8) + \frac{1}{2} \cdot U(0).$$

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Characterization of risk aversion – concavity

Definition. A function $G\colon \mathbb{R} \to \mathbb{R}$ is **concave** if for any $x, y \in \mathbb{R}$ and $\alpha \in (0,1)$,

$$G(\alpha x + (1-\alpha)y) \geq \alpha G(x) + (1-\alpha)G(y);$$

and **strictly concave** if for any $x, y \in \mathbb{R}$, , $x \neq y$, and $\alpha \in (0,1)$,

$$G(\alpha x + (1-\alpha)y) > \alpha G(x) + (1-\alpha)G(y).$$

Definition. A function $G\colon \mathbb{R} \to \mathbb{R}$ is **convex** if for any $x, y \in \mathbb{R}$ and $\alpha \in (0,1)$,

$$G(\alpha x + (1-\alpha)y) \leq \alpha G(x) + (1-\alpha)G(y).$$

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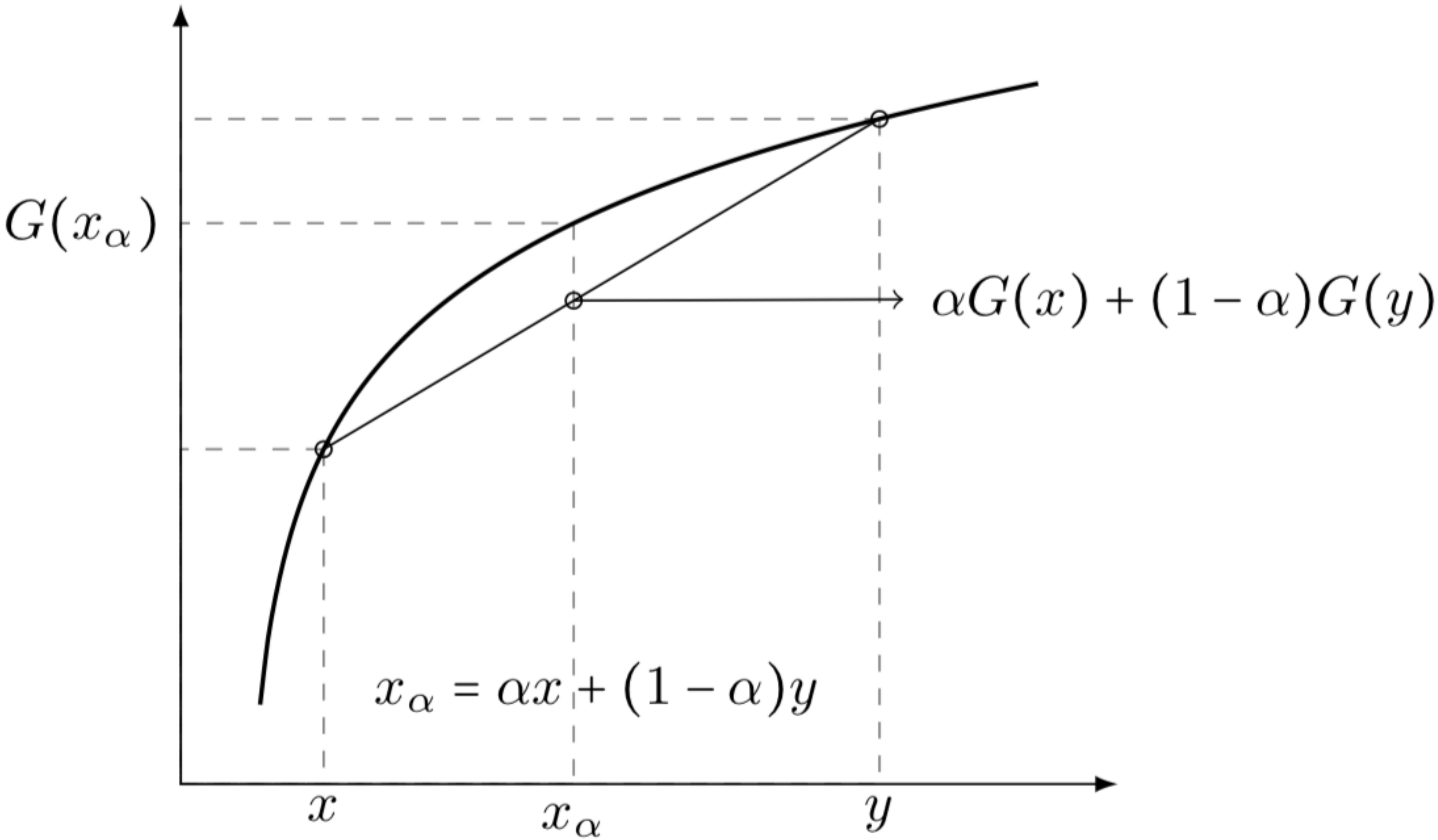
Concavity



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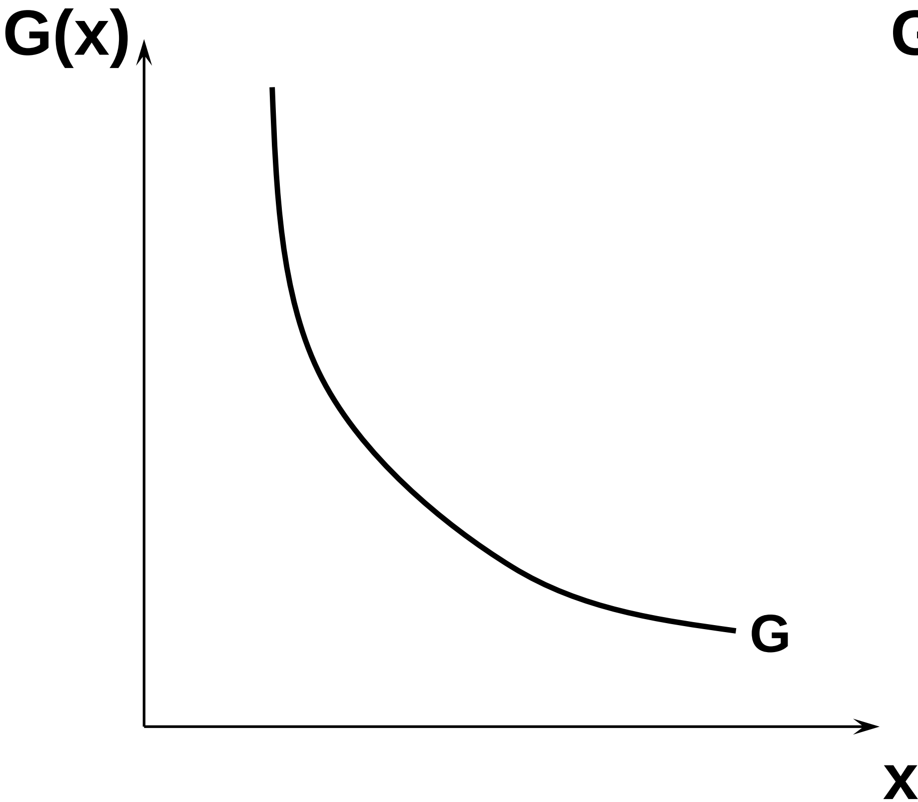
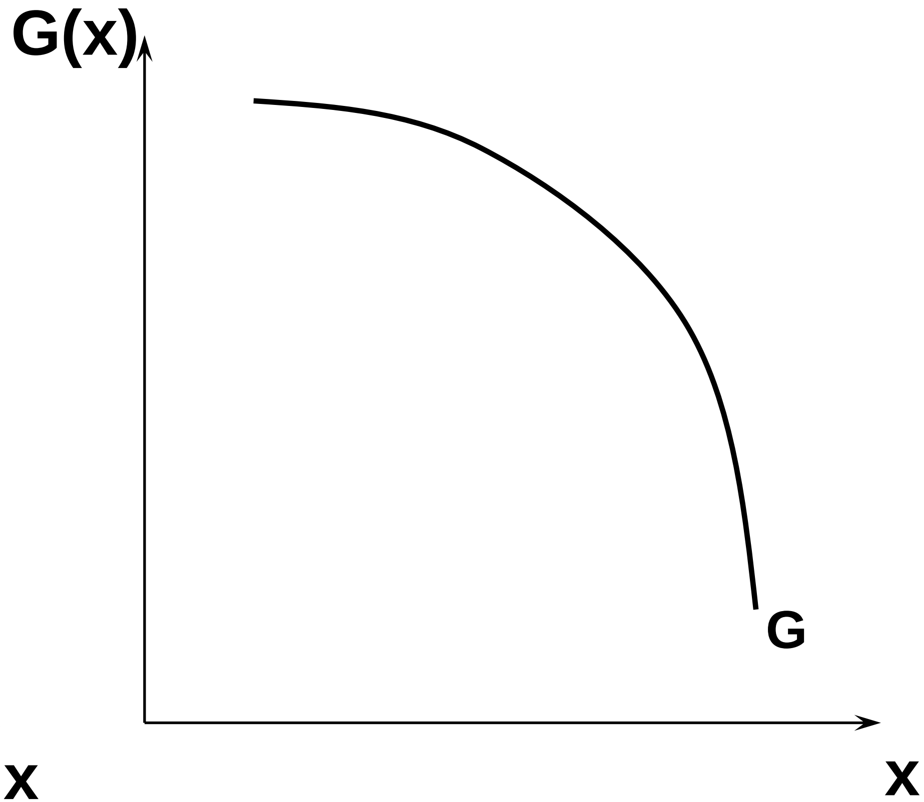
Concave and Convex Decreasing Functions

Convex:



Concave:



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Drawing an indifference curve

- If the function is twice differentiable, we can characterize concavity and convexity with the second derivative:

A function $G(x)$ is:

-concave if $G(x)'' \leq 0$

-strictly concave if $G(x)'' < 0$

-Convex if $G(x)'' \geq 0$

-Strictly convex if $G(x)'' > 0$

Characterization of risk aversion

Theorem. A Decision Maker (DM) that maximizes Expected Utility U(·) is risk averse if and only if $U\colon \mathbb{R} \rightarrow \mathbb{R}$ is concave.

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Relating risk aversion to the indifference curve

The indifference curve is the curve that gives us the combinations of consumption (i.e. x_1 *and* x_2) that provide the same level of Expected Utility

For any function of two variables (x_1 *and* x_2), the indifference curve can be obtained as a cut in the 3d plot

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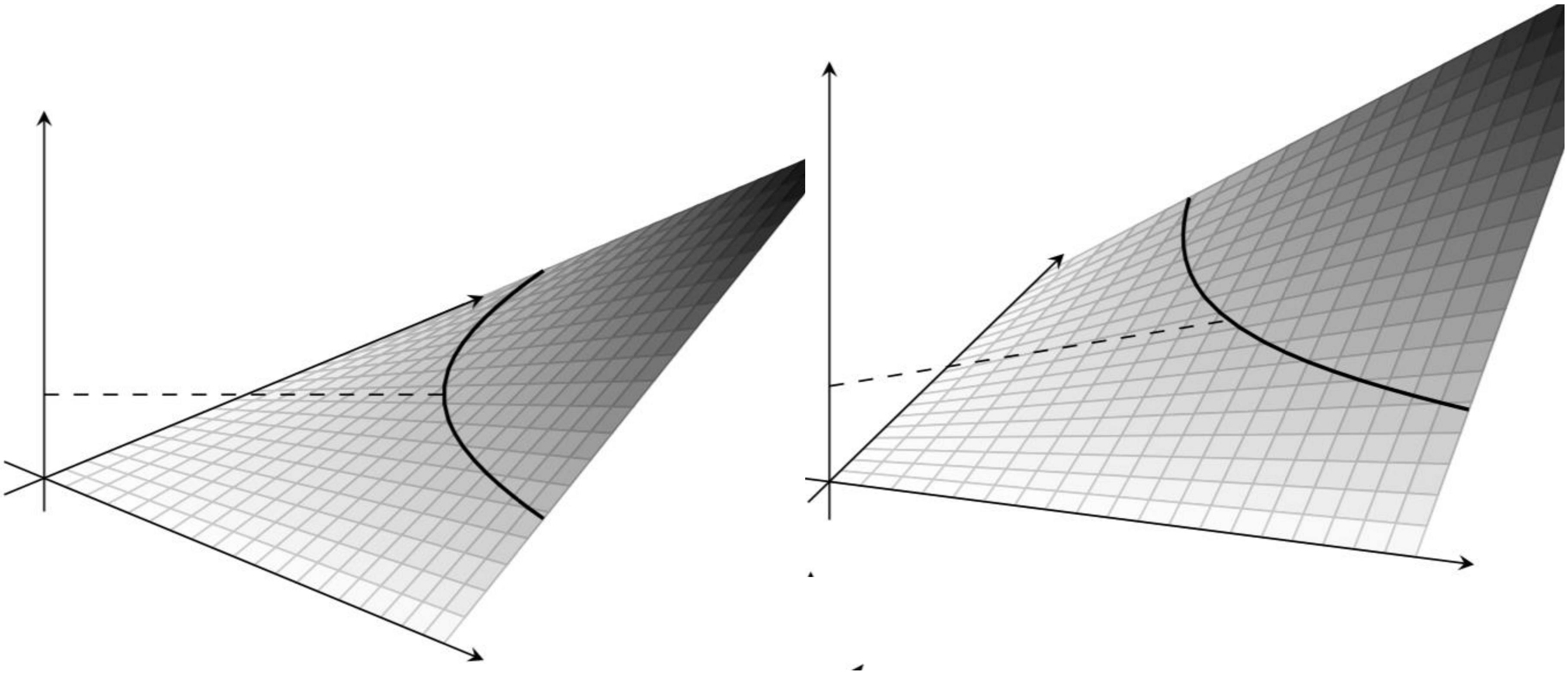
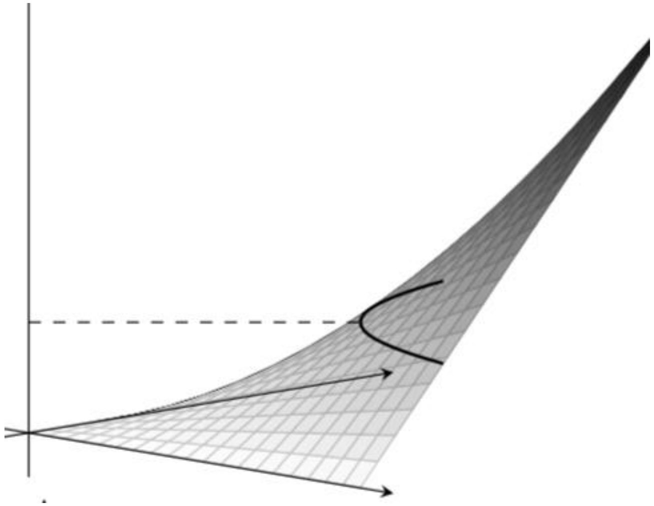
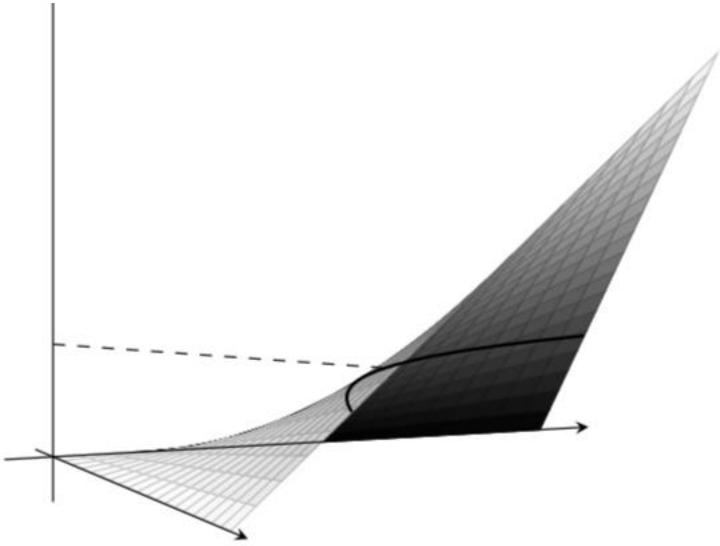
Indifference curve

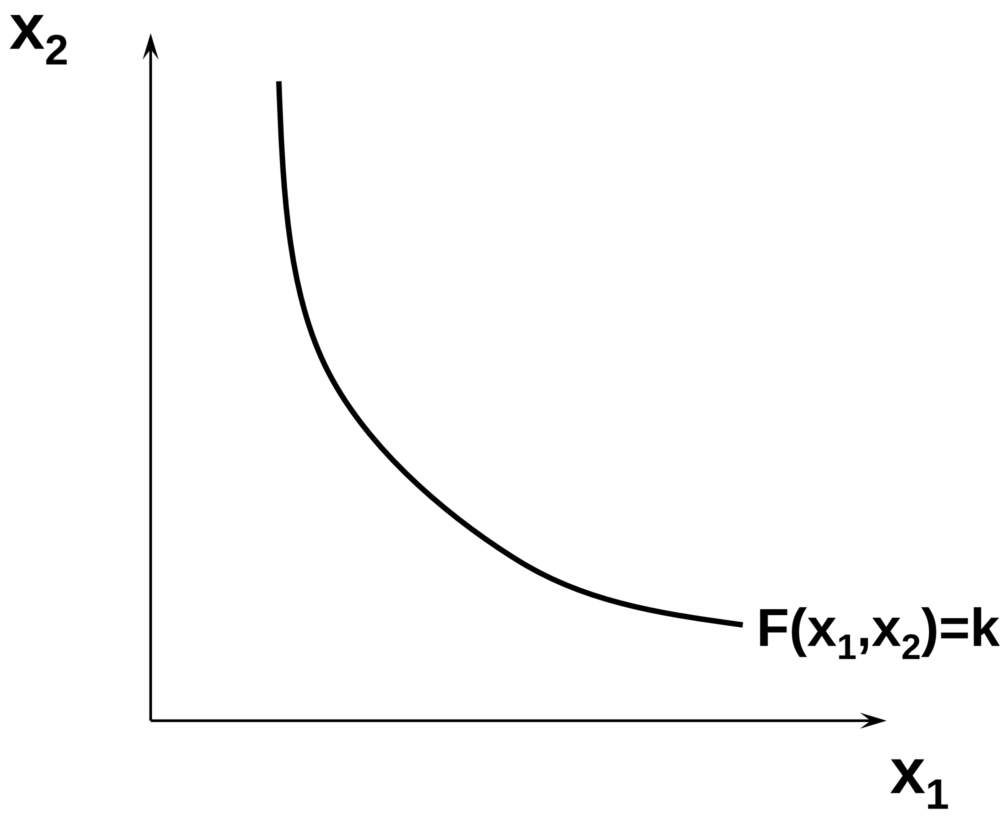


Indifference curve





Indifference curve



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Drawing an indifference curve

Are indifference curves decreasing or increasing?

$$EU = p_1 U(x_1) + p_2 U(x_2)$$

$$EU = p_1 U(x_1) + (1 - p_1) U(x_2)$$

$$dEU = 0 \Rightarrow p_1 U'(x_1) dx_1 + (1 - p_1) U'(x_2) dx_2$$

$$\frac{dx_2}{dx_1} = -\frac{p_1}{(1 - p_1)} * \frac{U'(x_1)}{U'(x_2)} = MRS$$

$$As\, U'(x) > 0 \Rightarrow \frac{dx_2}{dx_1} < 0 \Rightarrow decrea\sin g$$

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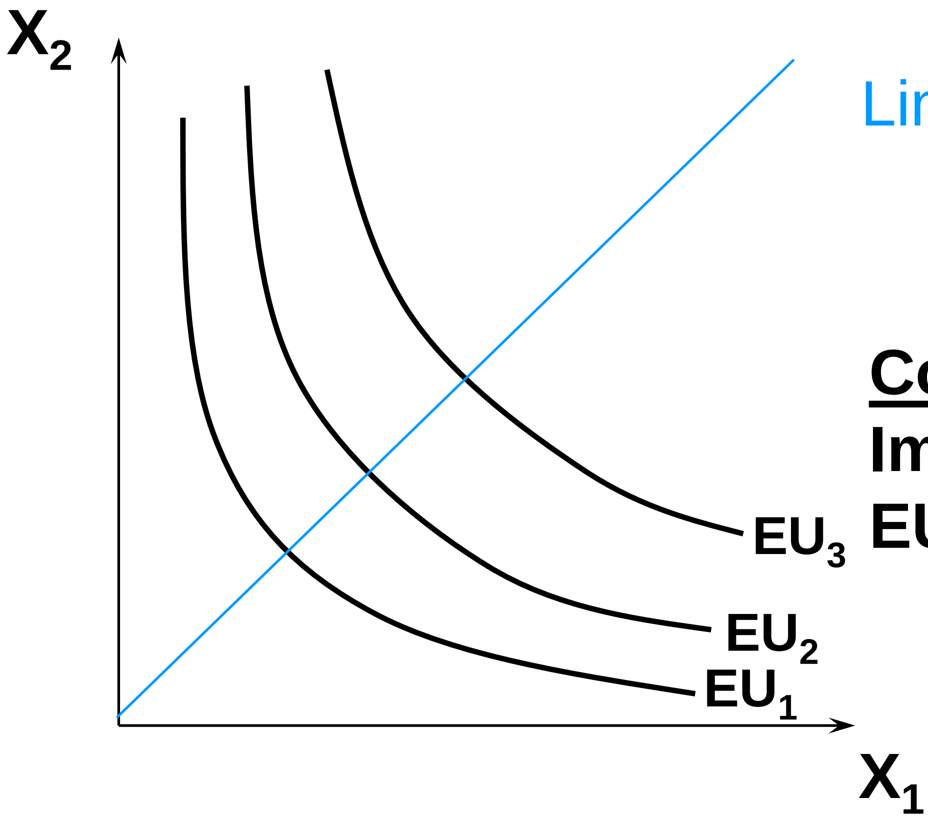
Drawing an indifference curve

- Ok, we know that the indifference curve will be decreasing
- What is their shape?
- For instance, we still do not know if they are convex or concave
- For the time being, let's assume that they are convex
    - This is related to the concavity of U(x), that is, U is concave if and only if the indifference curve is convex
- If we draw two indifferent curves, which one represents a higher level of utility?
- The one that is more to the right…

Drawing an indifference curve



Line of lotteries without risk

<u>Convex Indifference curves</u>

Important to understand that:

$EU_1 < EU_2 < EU_3$

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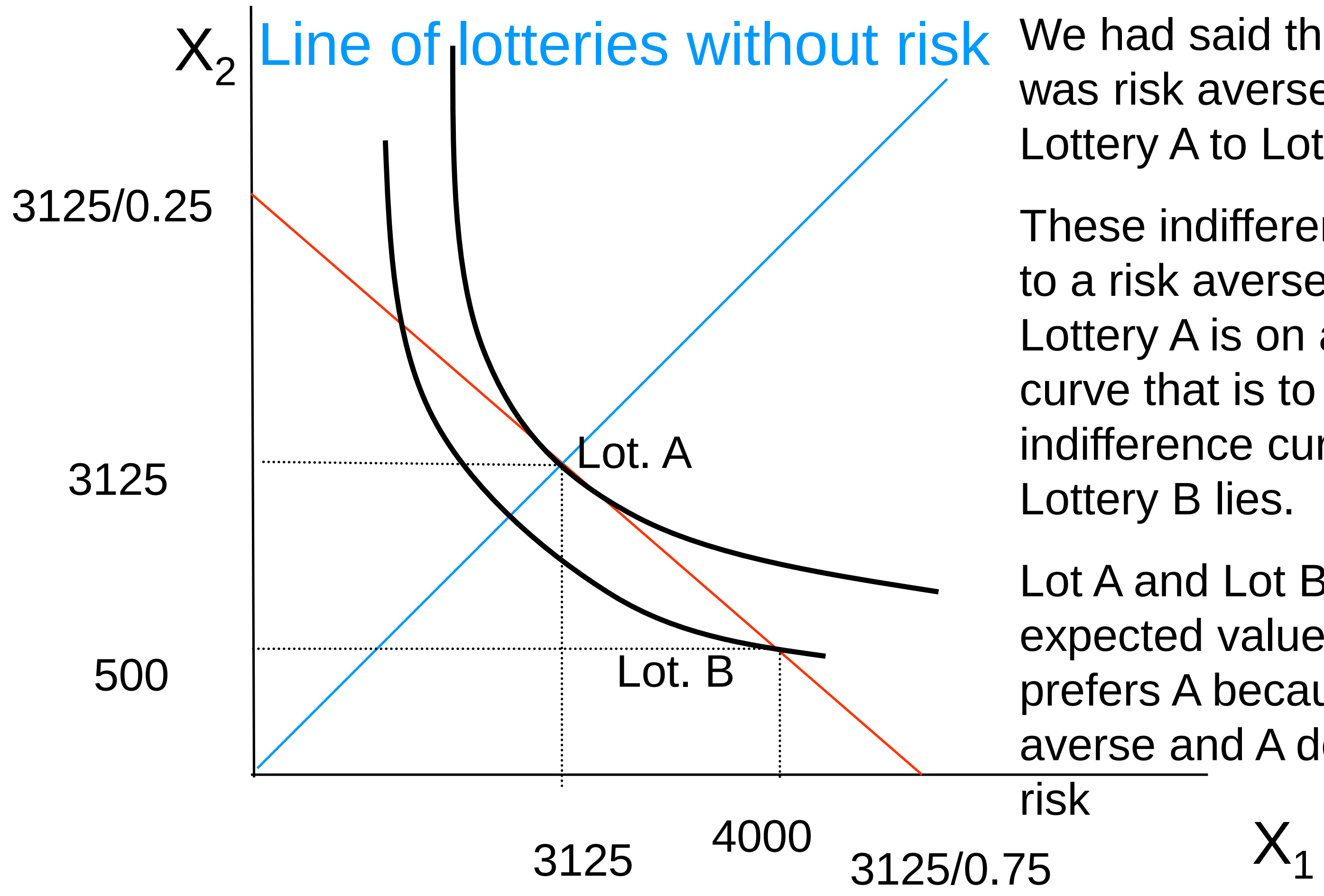
Indifference curve and risk aversion



We had said that if the individual was risk averse, he will prefer Lottery A to Lottery B.

These indifference curves belong to a risk averse individual as the Lottery A is on an indifference curve that is to the right of the indifference curve on which Lottery B lies.

Lot A and Lot B have the same expected value but the individual prefers A because he is risk averse and A does not involve risk

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Indifference curve and risk aversion

- We have just seen that if the indifference curves are convex then the individual is risk averse
- Could a risk averse individual have concave indifference curves?
- We say that if the indifference curve are concave then the individual is risk lover !!

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Risk aversion and the sign of U''(x)

$$\frac{dx_2}{dx_1} = -\frac{p_1}{(1-p_1)} * \frac{U'(x_1)}{U'(x_2)}$$

$$\frac{d^2x_2}{dx^2{}_1} = -\frac{p_1}{(1-p_1)} * \frac{U''(x_1)}{U'(x_2)}$$

- Convexity means that the second derivative is positive
- In order for this second derivative to be positive, we need that $U''(x)<0$
- A risk averse individual has utility function with $U''(x)<0$

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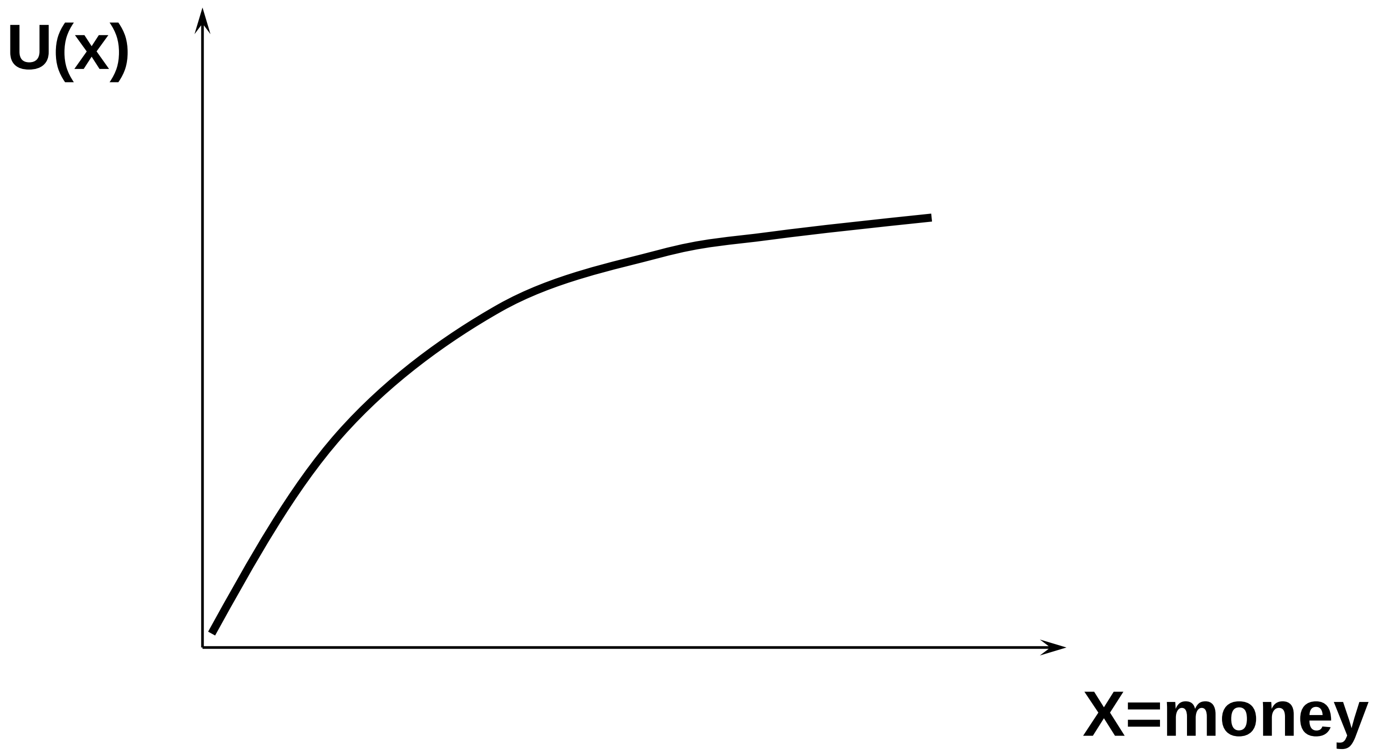
Concave utility function



- U'(x)>0, increasing
- U''(x)<0, strictly concave

Examples of concave utility functions

Examples of risk aversion

$$U(x) = \ln(x)$$

$$U(x) = \sqrt{x}$$

$$U(x) = x^a \quad where \quad 0 < a < 1$$

$$U(x) = -\exp(-a * x) \quad where \quad a > 0$$

$$U(x) = ax - bx^2, \text{ para } x \in \left[0, \frac{a}{2b}\right], a, b > 0$$

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Quadratic function

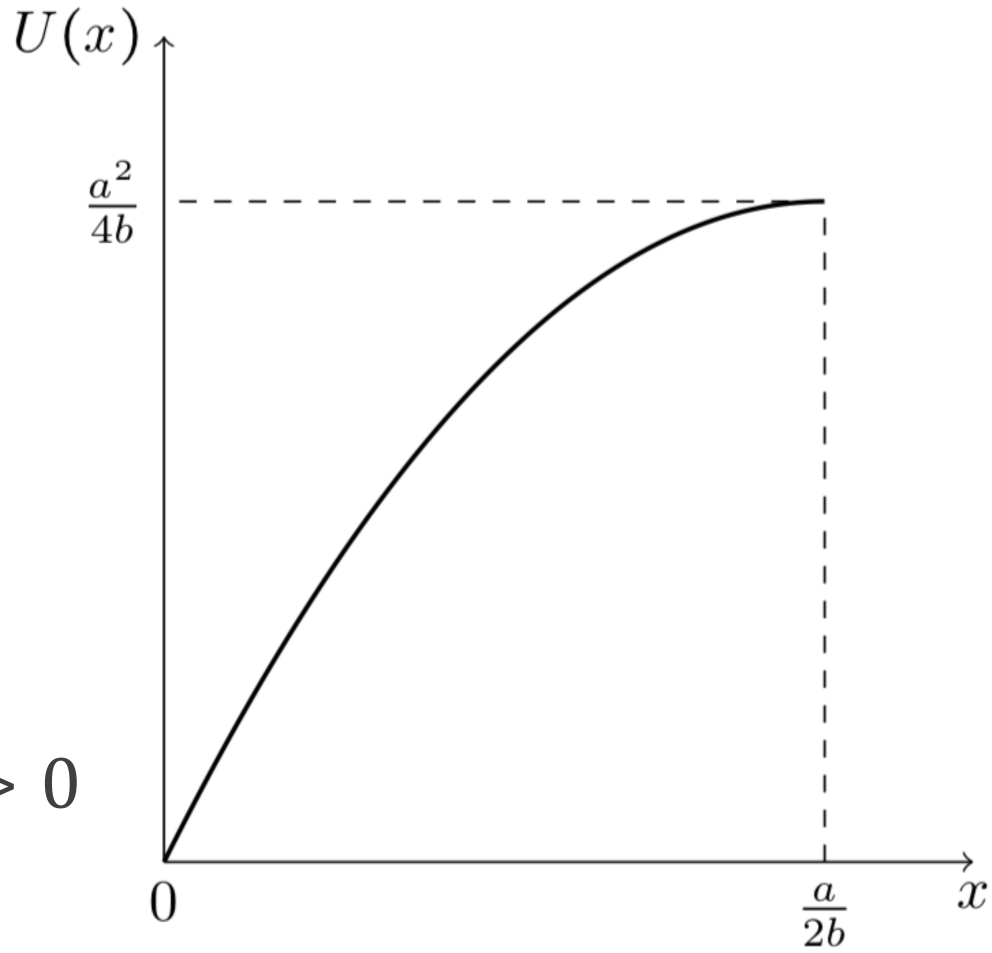
$$U(x) = ax - bx^2,$$

with a,b>0.

$$U'(x) = a - 2bx > 0,$$

$$if\ x < \frac{a}{2b}.$$

$$U''(x) = -2b < 0\,, if\ b > 0$$



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Risk neutrality

- Sometimes, we will assume that some individual is <u>risk neutral</u>
- Intuitively, this means that he does not like nor dislike risk
- Technically, it means that he or she is indifferent between a risky lottery and a risk free lottery as far as they have the same expected value
- Who could be like that?

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Risk neutrality

- Who can be risk neutral?
- If you play many times the risky lottery, you will get the expected value anyway
- So… you are indifferent between lotteries with the same expected value but different risk
- Individuals that play many times the same lottery behave as risk neutral
- Playing many times the risk lottery is similar to diversification

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Summary of classification

$$U''(X) < 0, \text{ stric concave U(X)} \Rightarrow \text{Risk averse}$$

$$U''(X) = 0, \text{ linear U(X)} \Rightarrow \text{Risk neutral}$$

$$U''(X) > 0, \text{ stric convex U(X)} \Rightarrow \text{Risk lover}$$

Strictly convex indiference curve $\Rightarrow$ Risk averse

Linear indifference curve $\Rightarrow$ Risk neutral

Strictly concave indifference curve $\Rightarrow$ Risk lover

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Measuring risk aversion

- The most commonly used risk aversion measure is the Arrow-Pratt coefficient of absolute risk aversion:

$$r(X) = -\frac{U''(X)}{U'(X)}$$

- For risk averse individuals, $U''(X) < 0$
  - $r(X)$ will be positive for risk averse individuals

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Risk aversion

- If utility is logarithmic in consumption

$$U(X) = ln\,(X)$$

where $X > 0$

- Pratt's risk aversion measure is

$U'(x) = \frac{1}{x}; U''(x) = -\frac{1}{x^2}$

$$r(X) = -\frac{U''(X)}{U'(X)} = \frac{1}{X}$$

Risk aversion decreases as wealth increases

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Risk aversion

- If utility is exponential

$$U(X) = -e^{-aX} = -exp\ (-aX)$$

where *a* is a positive constant

- Pratt's risk aversion measure is

$$r(X) = -\frac{U''(X)}{U'(X)} = \frac{a^2 e^{-aX}}{a e^{-aX}} = a$$

- Risk aversion is constant as wealth increases

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Willingness to pay for insurance

Consider a person with a current wealth of $ 100,000 who faces a 25% chance of losing his automobile worth $ 20,000

Suppose also that the utility function is

$$U(X) = ln\ (x)$$

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Insurance

The person's expected utility will be

$$E(U) = 0.75U(100{,}000) + 0.25U(80{,}000)$$

$$E(U) = 0.75\, ln(100{,}000) + 0.25\, ln(80{,}000)$$

$$E(U) = 11.45714$$

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Insurance

The individual will likely be willing to pay more than $ 5,000 to avoid the gamble. How much will he pay?

$$E(U) = U(100{,}000 - y) = ln(100{,}000 - y) = 11.45714$$

$$100{,}000 - y = e^{11.45714}$$

$$y = 5{,}426$$

The maximum premium he is willing to pay is $ 5,426

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Insurance

- If an agent buys an insurance policy at an actuarially fair premium then the insurance company will have zero expected profits
- Note: marketing and administration expenses are not included in the computation of the actuarially fair premium
- Let’s compute the expected profit of the insurance company in the previous example

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Insurance - Profits

$EP = 0.75p^{af} + 0.25(p^{af} - 20{,}000)$

Compute p^{af} such that $EP=0$.

This is p^{af}= **$** 5000

Notice, the actuarially fair premium is smaller than the maximum premium that the individual is willing to pay (**$** 5426). So there is room for the insurance company and the individual to trade and improve their profits/welfare

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Summary, so far

- The expected value is an adequate criterion to choose among lotteries if the individual is risk neutral
- However, it is not adequate if the individual dislikes risk (risk averse)
- If someone prefers to receive **$** B rather than playing a lottery in which expected value is **$** B then we say that the individual is <u>risk averse</u>
- If U(x) is the utility function then we always assume that U’(x)>0

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Summary, so far

- If an individual is risk averse then $U''(x)<0$, that is, the marginal utility is decreasing with money ($U'(x)$ is decreasing).
- If an individual is risk averse then his utility function, $U(x)$, is concave
- The Arrow-Pratt coefficient of absolute risk aversion is a standard measure of risk aversion
- The individual will insure if he is charged a fair premium

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Greening Energy Market and Finance

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